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Second order connections and stochastic horizontal lifts☆

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Abstract

In this paper we develop a theory of second order connections with a view towards the associated stochastic calculus. Connections in principal fiber bundles are defined as sections of the tangent space of second order differential operators. We prove existence and uniqueness of stochastic horizontal lifts for semimartingales with respect to these connections. Finally, the parallel transport along semimartingales on the base space is studied.

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1. Introduction

The purpose of this paper is to develop an absolute differential calculus of second order. We focus our attention on a special type of connections for Schwartz geometry, the 2-connections. They are objects analogous to the classical connections on fiber bundles, but specially adapted to the second order geometry. Naturally, each 2-connection has associated a stochastic horizontal lifting. We study the properties of 2-connections and their relationships with other type of second order connections in the context of the general theory. We remark that the 2-connections are implicit in the works of Meyer [27,28] and Schwartz [32].

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Our main motivation to study a second order geometry is its relevance in the stochastic calculus. In fact, let X be a continuous semimartingale in a manifold M, the Itô formula shows that Itô's differentials $\mathrm{d}X^i$ and $\mathrm{d}[X^i, X^j]$ (where (x^i) is a local chart and X^i the ith coordinate of X in this chart) behave under a change of coordinates as the coefficients of a second order tangent vector. The (purely formal) stochastic differential

$$\mathrm{d}_2 X_t = \mathrm{d}X_t^i \frac{\partial}{\partial x^i} + \frac{1}{2}\mathrm{d}[X^i, X^j]_t \frac{\partial^2}{\partial x^i \partial x^j}$$

is a linear differential operator on M, at X_t, of order at most two, with no constant term. Therefore, the tangent object to X_t is formally one of second order.

Let x be a point in a manifold M, the second order tangent space to M at x, denoted $\tau_x M$, is the vector space of all linear differential operators on M, at x, of order at most two, with no constant term. Formally, we have that $\mathrm{d}_2 X_t \in \tau_{X_t} M$. We develop a theory of connections adapted to this second order geometry and based on semimartingales instead of smooth curves.

Let $\pi : P \to M$ be a fiber bundle and $\mathbf{H} = \{H_p : p \in P\}$ a connection, where H_p is the horizontal lifting of $T_{\pi p}M$ to $T_p P$. The fundamental geometric construction associated with a connection is the horizontal lifting of smooth curves. Let γ be a smooth curve in M and define $\tilde{\gamma}$ to be a horizontal lifting of γ if it satisfies the differential equation $\mathrm{d}\tilde{\gamma}_t = H_{\tilde{\gamma}_t}\mathrm{d}\gamma_t$.

A family $\mathbf{H} = \{H_p : p \in P\}$ is said to be a 2-connection, if $H_p : \tau_{\pi p}M \to \tau_p P$ are Schwartz morphism (a special type of linear mapping, see below) such that $\pi_* \circ H_p = \mathrm{Id}_{\tau_{\pi p}M}$. Analogously to the case of smooth curves, we have an associated horizontal lifting of semimartingales. In fact, let X be a continuous M-valued semimartingale, we have that the P-valued semimartingale $\tilde{X}$ is a horizontal lifting of X if it satisfies the following stochastic differential equation:

$$\mathrm{d}_2 \tilde{X}_t = H_{\tilde{X}_t}\, \mathrm{d}_2 X_t.$$

The horizontal lifting of semimartingales with respect to a connection $\mathbf{H}$ were studied by Bismut [3], Shigekawa [33] and others (see Meyer [27], Ikeda and Watanabe [18], Schwartz [32] and Emery [11]), the first studies of Itô [19] and Dinkyn [10] are about horizontal lifting of Brownian motion. The present work is a formalization and extension of the articles mentioned.

In Section 3 we describe a formula which links the covariant derivative operator ∇, associated to a 2-connection, with the induced parallel transport over an L-diffusion (detailed definitions are presented in Section 3). This formula is similar to the usual one and reads

$$(//_{X_t}^{\nabla})^{-1}\varphi \circ X_t = (\text{local martingale})_t + \varphi(\pi(p)) + \int_0^t //_{X_s}^{-1}(\nabla_L \varphi \circ X_s)\,\mathrm{d}s, \tag{1}$$

where $//_{X_t}^{\nabla}$ is the parallel transport associated to ∇ of a section φ of the vector bundle over the L-diffusion X while the right hand side denotes covariant derivative of the section in the direction of the second order differential operator L.

The paper is organized as follows. In Section 2, we review some of the standard facts on Schwartz geometry (see for instance [11,13,27,28,32]) and introduce the fundamental notions of 2-connection and stochastic horizontal lifting. We prove that these connections are equivalent to the Ehresmann's holonomic connection of order two [14] and, by means of a construction of Meyer [27], we describe the Stratonovich prolongation of connections (in the usual sense) to 2-connections, in terms of jet theory. This prolongation is the basis of the Stratonovich transfer principle in stochastic calculus (see for instance Emery [12]). Finally, it is shown that the 2-connections of $\pi : P \to M$ are in bijection with pairs (Γ, Φ) where Γ is a connection in the usual sense and Φ is a section of $\mathrm{Ker}(\pi_*) \otimes \bigodot^2 T^*M \to P$. In Section 3, we study 2-connections in vector bundles and discuss different formulations of this notion. The L-connections of Liebermann [23,24], are a particular type of connection of order 2, we prove that every L-connection for a vector bundle E is induced by a 2-connection for the principal fiber bundle of the basis BE. We also show that the L-connections can be defined by operators of covariant derivatives encompassing the covariant derivatives of the classical connections in vector bundles. Finally, we obtain the formula (1) that links the operator of covariant derivative associated to a 2-connection, with the induced parallel transport.

2. Schwartz geometry and 2-connections

Throughout this paper all the geometrical objects like, manifolds, maps and functions will always be assumed to be smooth. As to manifolds, jet theory and stochastic differential geometry, we shall use freely concepts and notations of Emery [11], Kobayashi and Nomizu [20] and Kolar et al. [21].

Let x be a point in a manifold M. The second order tangent space to M at x, $\tau_x M$ is the vector space of all differential operators on M at x of order at most two without a constant term. Let (U, x^i) be a local coordinate system around x. Every $L \in \tau_x M$ can be write in a unique way as

$$L = a^i D_i + a^{ij} D_{ij},$$

where $a^{ij} = a^{ji}$, $D_i = \frac{\partial}{\partial x^i}$ and $D_{ij} = \frac{\partial}{\partial x^i \partial x^j}$ are differential operators at x (we shall use the convention of summing over repeated indices). The elements of $\tau_x M$ are called second order tangent vectors at x, the elements of the dual vector space $\tau_x^* M$ are called second order forms at x.

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ (respectively, $\tau^* M = \bigcup_{x \in M} \tau_x^* M$) is canonically endowed with a vector bundle structure over M, it is called the second order tangent fiber bundle (respectively, second order cotangent fiber bundle) of M.

Let $\phi : M \to N$ be a smooth map, and $L \in \tau_x M$. We have that $\phi_*(x)L \in \tau_{\phi(x)} N$, the differential of ϕ is given by

$$\phi_*(x)L(f) = L(f \circ \phi),$$

where $f \in \mathcal{C}^\infty(N)$. A covector $\theta \in \tau^*_{\phi(x)} N$ is pulled back into $\phi^*(x)\theta \in \tau_x^* M$ by

$$\langle \phi^*(x)\theta, L \rangle = \langle \theta, \phi_*(x)L \rangle,$$

where $L \in \tau_x M$.

Let L be a smooth section of τM. The square field operator associated to L, denoted by QL, is the symmetric tensor given by

$$QL(f, g) = \tfrac{1}{2}(L(fg) - fL(g) - gL(f)),$$

where $f, g \in C^\infty(M)$. We can consider $Q_x : \tau_x M \to T_x M \odot T_x M$ as the linear map defined by

$$Q_x(L = a^i D_i + a^{ij} D_{ij}) = a^{ij} D_i \odot D_j.$$

Push forward of second order vectors by smooth maps is related to the so called Schwartz morphisms between second order tangent vector bundles.

Definition 1. Let M and N be manifolds and take $x \in M$ and $y \in N$. A linear map $f : \tau_x M \to \tau_y N$ is called a Schwartz morphism if

(i) $f(T_x M) \subset T_y N$ and
(ii) for every $L \in \tau_x M$ we have that $Q(fL) = (f \otimes f)(QL)$.

For the convenience of the reader we prove a known result of Emery [11] which will be mentioned later.

Lemma 1. *A map $f : \tau_x M \to \tau_y N$ is a Schwartz morphism if and only if there exists a smooth map $\phi : M \to N$ with $\phi(x) = y$ such that $f = \phi_*(x)$.*

Proof. Let $\phi : M \to N$ be a smooth map with $\phi(x) = y$. Obviously, we have that $\phi_*(x)T_xM \subseteq T_yN$. Let $L \in \tau_x M$, we have that

$$Q(\phi_*(x)L)(g, h) = QL(g \circ \phi, h \circ \phi) = \phi_*(x) \otimes \phi_*(x)(QL)(g, h)$$

for all $g, h \in \mathcal{C}^\infty(N)$. We conclude that $\phi_*(x)$ is a Schwartz morphism. Conversely, let $f : \tau_x M \to \tau_y N$ be a Schwartz morphism and (U, x^i) and (V, y^λ) be local coordinate system around of x and y, respectively. Then f verifies

$$f(D_i) = f_i^\alpha D_\alpha \quad \text{and} \quad f(D_{ij}) = f_{ij}^\alpha D_\alpha + \tfrac{1}{2}(f_i^\alpha f_j^\beta + f_j^\alpha f_i^\beta)D_{\alpha\beta}.$$

Since there exists a smooth map $\phi : M \to N$ such that

$$D_i y^\alpha \circ \phi(x) = f_i^\alpha, \qquad D_{ij} y^\alpha \circ \phi(x) = f_{ij}^\alpha$$

we have that the proposition holds. □

Definition 2. Let $\pi : P \to M$ be a fiber bundle. A family of Schwartz morphisms $\mathbf{H} = \{H_p : p \in P\}$ is called a 2-connection if

(1) $H_p : \tau_{\pi(p)} M \to \tau_p P$,
(2) $\pi_* \circ H_p = \mathrm{Id}_{\tau_{\pi(p)} M}$, and
(3) the map $p \to H_p L$ is a smooth section of τP, where L is an arbitrary smooth section of τM.
Whenever $P = P(M, G)$ is a principal fiber bundle, we say that $\mathbf{H}$ is a principal 2-connection if it also verifies the invariance under G, namely:
(4) $H_{pg} = R_{g_*} H_p$ for all $p \in P$ and $g \in G$ where R_g stands for the right action of G in P.

Next, we describe these connections in local coordinates. Let (x^λ, y^i) and (x^λ) be local charts for P and M, respectively, then a 2-connection $\mathbf{H}$ for $\pi : P \to M$ can be written in a unique way as

$$H_p(D_\lambda) = D_\lambda + a_\lambda^i D_i, \qquad H_p(D_{\lambda\mu}) = D_{\lambda\mu} + a_{\lambda\mu}^i D_i + a_{\lambda\mu}^{ij} D_{ij} + 2a_{\lambda\mu}^{i\nu} D_{i\nu},$$

where

$$a_{\lambda\mu}^{ij} = \tfrac{1}{2}(a_\lambda^i a_\mu^j + a_\mu^i a_\lambda^j), \qquad a_{\lambda\mu}^{i\nu} = \tfrac{1}{2}(a_\lambda^i \delta_\mu^\nu + a_\mu^i \delta_\lambda^\nu).$$

These equalities guarantee that H_p is a Schwartz morphism.

We consider also a filtered probability space $(\Omega, F, \mathbf{P}, (F_t)_{t\geq 0})$ satisfying the usual conditions (see for example [11,13,30]). We always assume that the semimartingales have continuous paths and are adapted to the same filtered probability space. Let $\pi : P \to M$ be a fiber bundle and $\mathbf{H} = \{H_p : p \in P\}$ be a 2-connection.

Definition 3. Let X be a continuous M-valued semimartingale and ρ a stopping time. A stochastic horizontal lifting up to ρ of X with respect to the 2-connection $\mathbf{H}$, starting at $p \in P$, is a P-valued semimartingale $\tilde{X}_t$ with lifetime ρ satisfying:

(i) $\tilde{X}_0 = p$,
(ii) $\pi \tilde{X}_t(\omega) = X_t(\omega)$ a.s. and
(iii) $\int_0^t \langle \theta, \mathrm{d}_2 \tilde{X} \rangle = 0$ for all t such that $0 \leq t \leq \rho$ a.s. Here θ is an arbitrary section of $\tau^* P$ such that $\theta \circ \mathbf{H} = 0$.

We say that $\tilde{X}$ is a stochastic horizontal lift (SHL, for short) of X to P up to ρ.

The existence of SHL is established by our next theorem.

Theorem 1. *Let $\pi : P \to M$ be a fiber bundle, $\mathbf{H}$ a 2-connection and $p \in P$. Then for any M-valued semimartingale $X = (X_t)$ such that $X_0 = \pi(p)$, there exists a predictable stopping time $\rho > 0$ and a stochastic horizontal lifting $\tilde{X}$ up to ρ of X starting at p. Moreover, ρ and $\tilde{X}$ satisfy the following property of uniqueness: if ρ' is a predictable stooping time and X' is another SHL of X starting at p up to ρ' then $\rho' \leq \rho$ and $X' = \tilde{X}$ in $[0, \rho')$.*

Proof. Let $\tilde{X}_t$ be the maximal solution of the following stochastic differential equation:

$$\mathrm{d}_2 \tilde{X}_t = \mathbf{H}(\tilde{X}_t)\, \mathrm{d}_2 X_t, \qquad \tilde{X}_0 = p.$$

Existence of this solution is guaranteed because the equation has locally Lipschitz coefficients (see [30]). Obviously $\pi \tilde{X}_t = X_t$ and if θ is a 2-form on P such that $\theta \circ \mathbf{H} = 0$, it follows that

$$\int_0^t \langle \theta, \mathrm{d}_2 \tilde{X} \rangle = \int_0^t \langle \mathbf{H}^*(\tilde{X})\theta, \mathrm{d}_2 X \rangle = \int_0^t \langle \theta \circ \mathbf{H}(\tilde{X}), \mathrm{d}_2 X \rangle = 0.$$

Then $\tilde{X}$ is a stochastic horizontal lift of X. The uniqueness of the horizontal lifting, follows from the uniqueness of solution of stochastic differential equations with locally Lipschitz coefficients, up to a certain stopping time. □

Naturally, every principal 2-connection $\mathbf{H}$ for a principal fiber bundle $P(M, G)$ induces a unique connection in the usual sense by restriction to the tangent space $\mathbf{H}_R = \{(H_R)_p = H_p | T_{\pi p} M : p \in P\}$. Conversely, every connection $\mathbf{H}$ for a fiber bundle $\pi : P \to M$ has a canonical prolongation to a 2-connection $\mathbf{H}^S$ that satisfies $H_p^S | T_{\pi(p)} = H_p$ for all $p \in P$ (see [27]). This prolongation is the Stratonovich prolongation and is characterized by

$$H_p^S\{X, Y\} = \{HX, HY\}_p$$

for all local vector fields X, Y of M. Where $\{A, B\} = \frac{1}{2}(AB + BA)$ is the skew-commutator of the local vector fields A and B.

The next lemma shows that in the case of principal fiber bundles, the Stratonovich prolongation of a connection (in the usual sense) is a principal 2-connection.

Lemma 2. *Let $P(M, G)$ be a principal fiber bundle and $\mathbf{H}$ be a connection in the usual sense. Then $\mathbf{H}^S$ is a principal 2-connection.*

Proof. We only need to prove that, $R_{g*} \circ H_p^S = H_{pg}^S$ for all $p \in P$ and $g \in G$.

Let X and Y be local vector fields in a neighborhood of $\pi(p) \in M$. Since

$$R_{g*} \circ H_p^S(X) = R_{g*} \circ H_p(X) = H_{pg}(X) = H_{pg}^S(X)$$

and

$$R_{g*} \circ H_p^S(\{X, Y\}) = R_{g*}(\{HX, HY\}_p) = \{HX, HY\}_{pg} = H_{pg}^S(\{X, Y\}).$$

We conclude that $R_{g*} \circ H_p^S = H_{pg}^S$. □

The following construction gives another prolongation of connections to principal 2-connections (see [5] for more details). Let $P(M, G)$ be a principal fiber bundle and ∇ be a G-invariant operator of covariant derivative of P such that $\nabla_A B$ is vertical for any vertical field A. Let $\mathbf{H} = \{H_p : p \in P\}$ be a connection for $P(M, G)$ in the usual sense. Then there exists an unique principal 2-connection $\mathbf{H}^\nabla = \{H_p^\nabla : p \in P\}$ such that

$$H^\nabla\{X, Y\} = \{HX, HY\} - \omega^H(\nabla_{HX} HY + \nabla_{HY} HX)^*, \tag{2}$$

where X and Y are local fields of M, ω^H is the connection form associated with $\mathbf{H}$ and $*$ is the homomorphism defined by the right action of G on P.

The following result ensures that in the case of a principal fiber bundle the lifetime of an M-valued semimartingale and its horizontal lifting are the same.

Theorem 2. *Let $P(M, G)$ be a principal fiber bundle and $\mathbf{H}$ be a principal 2-connection. Let X be an M-valued semimartingale and $\tilde{X}$ a SHL of X with respect to $\mathbf{H}$. Then X and $\tilde{X}$ have the same lifetime.*

Proof. Let $\mathbf{H}_R$ be the connection (in the usual sense) induced by $\mathbf{H}$. We can assume without loss of generality that the lifetime of X is ∞. We recall that Shigekawa [33] proved that X^S, the stochastic horizontal lifting in (the sense of Stratonovich) of X with respect to the connection $\mathbf{H}_R$, also has time life ∞ and that X^S is the SHL of X with respect to $\mathbf{H}_R^S$. We define $b(\mathbf{H})$ the $\mathcal{G}$-valued section of τP^*, by

$$b(\mathbf{H})_p = (p_*(e))^{-1} \circ (H_p - (H_R)_p^S) \circ \pi_*(p),$$

where we consider $p \in P$ as the mapping of G into P given by $p(g) = pg$. Now, we consider the G-valued semimartingale $g = \epsilon(\int b(\mathbf{H})\, \mathrm{d}_2 X^S)$, where ϵ is the left stochastic exponential (see [17]). We note that g is a finite variation process with time life ∞. We claim that $X^S g$ is the SHL of X respect to $\mathbf{H}$. In fact, we have that

$$\begin{aligned}
\mathrm{d}_2(X^S g) &= (\mathrm{d}_2 X^S)g + X^S\, \mathrm{d}_2 g = ((H_R^S)_{X^S}\, \mathrm{d}_2 X)g + X^S g\, \mathrm{d}_2\epsilon\left(\int b(\mathbf{H})\, \mathrm{d}_2 X^S\right)\\
&= (H_R^S)_{X^S g}\, \mathrm{d}_2 X + X^S g b(\mathbf{H})\, \mathrm{d}_2 X^S = (H_R^S)_{X^S g}\, \mathrm{d}_2 X + X^S g b(\mathbf{H})(R_g)_*\, \mathrm{d}_2 X^S\\
&= (H_R^S)_{X^S g}\, \mathrm{d}_2 X + (H - (H_R)^S)_{X^S g} \circ \pi_*((R_g)_*\, \mathrm{d}_2 X^S)\\
&= (H_R^S)_{X^S g}\, \mathrm{d}_2 X + (H - (H_R)^S)_{X^S g}\, \mathrm{d}_2 X = H_{X^S g}\, \mathrm{d}_2 X. \quad \square
\end{aligned}$$

Example 1. Let M be a manifold equipped with an operator of covariant derivative without torsion ∇. Let BM be the principal fiber bundle of frames of M, it is known that ∇^C the complete lift of ∇ is an $GL(n)$-invariant operator of covariant derivative on BM with projection ∇ and that $\nabla_A^C B$ is vertical for any vertical field A (see [8, pp. 66 and 95]). Let $\mathbf{H} = \{H_p : p \in BM\}$ be the connection for BM associated with ∇. According to the comments after Lemma 2, we have that there exists an unique principal 2-connection $\mathbf{H}^C := \{H_p^C : p \in BM\}$ (prolongation of $\mathbf{H}$) satisfying (2). The stochastic horizontal lift with respect to $\mathbf{H}^C$ is the deformed (damped, geodesic, Dohrn-Guerra) parallel transport to BM (see [2,9,26,28]). The following result has been proved in [5].

Let X_t be the solution of the Stratonovich equation on M starting at $x \in M$,

$$\mathrm{d}X_t = A_0(X_t)\, \mathrm{d}t + \sum_{i=1}^n A_i(X_t) \circ \mathrm{d}B_t^i,$$

where $A_0, \ldots, A_n$ are vector fields on M and $(B_t^1, \ldots, B_t^n)$ a Brownian motion on $\mathbb{R}^n$. Then Y_t the SHL starting at $p \in BM$ $(\pi(p) = x)$ with respect to $\mathbf{H}^C$ of X is solution of the Stratonovich equation on BM:

$$\mathrm{d}Y_t = \left(HA_0 - \frac{1}{2}\sum_{i=1}^{n} \widetilde{R(-, A_i)}A_i \right)(Y_t)\,\mathrm{d}t + \sum_{i=1}^{n} HA_i(Y_t) \circ \mathrm{d}B_t^i,$$

where R is the curvature tensor, $R(-, X)Y(Z) := R(Z, X)Y$ and $\widetilde{R(-, X)}Y(p) := (p^{-1}R(-, X)Yp)^*(p)$. In the case that M is a Riemannian manifold, ∇ the Riemannian operator of covariant derivative and Δ the associated Laplacian, we have that

$$H_p^{\nabla}(\Delta) = \sum_{i=1}^{n} E_i^2(p) - \sum_{i=1}^{n} \widetilde{R(-, E_i)}E_i(p),$$

where $E_i(p) := H_p(pe_i)$ are the standard horizontal fields associated to ∇.

We will show that every 2-connection induces a holonomic connection of order two in the Ehresmann sense (see [14]). This type of connection is given by a section of $p_G : J^2P/G \to M$ where J^2P/G is the quotient of J^2P by the natural action of G in J^2P and p_G is induced by the source projection $\alpha : J^2 \to M$.

Before proceeding further, we need some concepts and terminology of jet theory (see [21,22,29,34]). Let $\pi : P \to M$ be a fiber bundle and s, s' be two local sections. We say that s and s' are 2-equivalent at $x \in M$ if they satisfy:

- $s(x) = s'(x)$,
- $D_i s(x) = D_i s'(x)$ for $1 \le i \le n$,
- $D_{ij} s(x) = D_{ij} s'(x)$ for $1 \le i, j \le n$.

It follows that we have an equivalence relation on the set of local sections of $\pi : P \to M$. The equivalence class of a section s is called the 2-jet of s at x and is denoted by $j_x^2 s$. The point x is called the source of the jet and $s(x) \in P$ is called its target.

Let J^2P_x be the set of the 2-jets at x of local sections of $\pi : P \to M$ and set $J^2P = \bigcup_{x\in M} J^2P_x$, we says that J^2P is the set of holonomic 2-jets of P. Note that J^2P is a fiber bundle over M by the source projection $\alpha : J^2P \to M$, $\alpha(j_x^2 s) = x$ and also it is a fiber bundle over P by the target projection $\beta : J^2P \to P$, $\beta(j_x^2 s) = s(x)$.

Let $\pi : P \to M$ be a fiber bundle. We define a mapping Γ of the set of 2-connections to smooth sections of $\beta : J^2P \to P$ as follows. Let $\mathbf{H} = \{H_p : p \in M\}$ be a 2-connection. For each $p \in P$, there exists a smooth section $s : M \to P$ such that $s(\pi(p)) = p$ and $H_p = s_*(\pi(p))$. We set

$$\Gamma(\mathbf{H})(p) = j_{\pi(p)}^2 s.$$

It is clear that $\Gamma(\mathbf{H})$ is a section of $\beta : J^2P \to P$, moreover an easy computation shows that Γ is bijective.

Now, we specialize Γ to the case of principal fiber bundles. Let $P(M, G)$ be a principal fiber bundle and $\mathbf{H} = \{H_p : p \in M\}$ a principal 2-connection. For each $x \in M$ we take $p \in P$ such that $\pi(p) = x$, then there exists a smooth section $s : M \to P$ such that $s(x) = p$ and $H_p = s_*(x)$. We define

$$\Gamma(\mathbf{H})(x) = j_x^2 sG,$$

where $j_x^2 sG$ denote the equivalence class for the induced right action of G on J^2P. In order to see that this definition is independent of the representative, take p and q in P such that $\pi(p) = \pi(q) = x$ i.e. there exists $g \in G$ such that $q = pg$. Else

$$H_q = H_{pg} = R_{g_*}(p)H_p = R_{g_*}(p)s_*(x) = (R_g \circ s)_*(x).$$

We have that $\tilde{s} = R_g \circ s$ is a smooth section such that $\tilde{s}(x) = q$, $H_q = \tilde{s}_*(x)$ and

$$j_x^2(\tilde{s}) = j_x^2(R_g \circ s) = (j_x^2 s)g.$$

We conclude that the definition is independent of the representative and that $\Gamma(\mathbf{H})$ is a smooth section of $p_G : J_2P/G \to M$.

Proposition 1. *Let $P(M, G)$ be a principal fiber bundle. The mapping Γ is bijective between the set of 2-connections of $P(M, G)$ and the set of smooth sections of $p_G : J_2P/G \to M$.*

Proof. Let Ψ be a smooth section of $p_G : J_2P/G \to M$ and take $p \in P$. We can take a section s of $P(M, G)$ such that $\Psi(\pi(p)) = j^2_{\pi(p)}sG$ and $s(\pi(p)) = p$. We write H_p^Ψ for $s_*(\pi(p))$ and will prove that $\mathbf{H}^\Psi = \{H_p^\Psi : p \in P\}$ is a 2-connection for $P(M, G)$. In fact, we have that H_p^Ψ is a Schwartz morphism:

$$\pi_* \circ H_p^\Psi = \pi_* \circ s_*(\pi(p)) = (\pi \circ s)_*(\pi(p)) = \mathrm{Id}_{\tau_{\pi(p)}M}$$

and

$$H_{pg}^\Psi = (R_g \circ s)_*(\pi(pg)) = R_{g_*}(p) \circ s_*(\pi(p)) = R_{g_*}(p) \circ H_p^\Psi$$

for all $g \in G$. The smoothness of $\mathbf{H}^\Psi$ follows immediately from the smoothness of Ψ. Finally, we observe that $\Gamma(H^\Psi) = \Psi$. □

Our next goal is to give a characterization of the Stratonovich prolongation, in terms of jet theory. For the convenience of the reader we present the necessary material of semiholonomic jets of order two.

Let $\pi : P \to M$ and $\theta : Q \to M$ be fiber bundles and $\phi : P \to Q$ a morphism of fiber bundles (this is $\theta \circ \phi(p) = \pi(p)$ for all $p \in P$). Let $j^1\phi : J^1P \to J^1Q$ be the unique morphism of fiber bundles over M such that $j^1\phi(j^1s) = j^1(\phi \circ s)$ for all local section s of $\pi : P \to M$ (see for instance [16,29] or [21]).

The set $\overline{J^2P}_x$ of 2-semiholonomic jets of $\pi : P \to M$ at $x \in M$ is given by

$$\{j_x^1 s : s \text{ is a local section of } \alpha : J^1P \to M \text{ at } x \text{ such that } s(x) = j_x^1\beta \circ s\}.$$

We have that $\alpha : \overline{J^2P} = \bigcup_{x\in M} \overline{J^2P} \to M$ is a fiber bundle called the bundle of 2-semiholonomics jets of P, for the proofs we refer the reader to [29,24].

Let (x^λ) and (x^λ, y^i) be local charts of M and P, respectively. We have the induced local charts $(x^\lambda, y^i, y^i_\lambda)$, $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu})$ and $(x^\lambda, y^i, y^i_\lambda, v^i_{\lambda\mu})$ for J^1P, J^2P and $\overline{J^2P}$, where y^i_λ, $y^i_{\lambda\mu}$ and $v^i_{\lambda\mu}$ are given by

$$y^i_\lambda(j_x^1 s) = D_\lambda y^i(s(x)), \qquad y^i_{\lambda\mu}(j_x^2 s) = D_{\lambda\mu} y^i(s(x)), \qquad v^i_{\lambda\mu}(j_x^1 \bar{s}) = D_\mu y^i_\lambda(\bar{s}(x)),$$

where s is a local section of $\pi : P \to M$ and $\bar{s}$ is a local section of $\alpha : J^1P \to M$.

The morphism of symmetrization $\mathrm{Sym} : \overline{J^2P} \to J^2P$ is given in the induced local charts for $\overline{J^2P}$ and J^2P, respectively, by

$$\mathrm{Sym}(x^\lambda, y^i, y^i_\lambda, v^i_{\lambda\mu}) = (x^\lambda, y^i, y^i_\lambda, \tfrac{1}{2}(v^i_{\lambda\mu} + v^i_{\mu\lambda})).$$

Now, we consider the application $*$ from sections of $\beta : J^1P \to P$ to section of $\beta : J^2P \to P$, given by

$$\varphi^* = \mathrm{Sym} \circ j^1\varphi \circ \varphi,$$

where φ is a section of $\beta : J^1P \to P$.

It is clear that $*$ is a prolongation. In fact, $\rho_1^2 \circ \varphi^* = \varphi$, where $\rho_1^2 : J^2P \to J^1P$ is defined by $\rho_1^2(j_x^2 s) = j_x^1 s$.

We recall that in the jet approach to connections in fiber bundles (see for instance Kolar et al. [21]), a connection Λ of the fiber bundle $\pi : P \to M$ is a section of $\beta : J^1P \to P$ and that a section of $\beta : J^2P \to P$ is a second order holonomic connection (see Cabras and Kolar [4] and Ehresmann [14]). In this context $*$ is a prolongation of connections of $\pi : P \to M$ to second order holonomic connections.

The following result is an intrinsic characterization of the Stratonovich prolongation.

Proposition 2. *Let $\pi : P \to M$ be a fiber bundle and $\mathbf{H}$ a connection. Then*

$$\Gamma(\mathbf{H}^S) = \Gamma(\mathbf{H})^*$$

Proof. Let $(x^\lambda, y^i, y^i_\lambda)$ and $(x^\lambda, y^i, y^i_\lambda, v^i_{\lambda\mu})$ be the standard local charts for J^1P and $\overline{J^2P}$, respectively, where (x^λ) and (x^λ, y^i) are local charts for M and P. Let φ be a section of $\beta : J_1P \to P$. In this local charts φ is written as

$$\varphi : (x^\lambda, y^i) \to (x^\lambda, y^i, \varphi^i_\lambda(x^\beta, y^j))$$

thus

$$j^1\varphi : (x^\lambda, y^i, y^i_\lambda) \to ((x^\lambda, y^i, \varphi^i_\lambda), y^i_\lambda, D_\zeta\varphi^i_\eta + y^j_\zeta D_j\varphi^i_\eta)$$

and

$$j^1\varphi \circ \varphi : (x^\lambda, y^i) \to ((x^\lambda, y^i, \varphi^i_\lambda), D_\zeta\varphi^i_\eta + \varphi^j_\zeta D_j\varphi^i_\eta).$$

Thus $\varphi^* = \mathrm{Sym} \circ j^1\varphi \circ \varphi$ is written as

$$\varphi^* : (x^\lambda, y^i) \to (x^\lambda, y^i, \varphi^i_\lambda, \tfrac{1}{2}(D_\zeta\varphi^i_\eta + D_\eta\varphi^i_\zeta + \varphi^j_\zeta D_j\varphi^i_\eta + \varphi^j_\eta D_j\varphi^i_\zeta)).$$

Let $\mathbf{H} = \{H_p : p \in P\}$ be a connection, (x^λ) and (x^λ, y^i) be local charts for M and P, respectively. We have that in these local charts $H_p(D_\lambda) = D_\lambda + a^i_\lambda D_i$ and

$$H^S_p(D_\lambda) = D_\lambda + a^i_\lambda D_i, \qquad H^S_p(D_{\lambda\mu}) = D_{\lambda\mu} + a^{ij}_{\lambda\mu} D_{ij} + a^i_{\lambda\mu} D_i + 2a^{i\nu}_{\lambda\mu} D_{i\nu},$$

where

$$a^{ij}_{\lambda\mu} = \tfrac{1}{2}(a^i_\lambda a^j_\mu + a^i_\mu a^j_\lambda), \qquad a^{i\nu}_{\lambda\mu} = \tfrac{1}{2}(a^i_\lambda \delta^\nu_\mu + a^i_\mu \delta^\nu_\lambda),$$

$$a^i_{\lambda\mu} = \tfrac{1}{2}(a^j_\lambda D_j a^i_\mu + a^j_\mu D_j a^i_\lambda + D_\lambda a^i_\mu + D_\mu a^i_\lambda).$$

It follows that

$$\Gamma(\mathbf{H}) : (x^\lambda, y^i) \to (x^\lambda, y^i, a^i_\lambda)$$

and

$$\Gamma(\mathbf{H}^S) : (x^\lambda, y^i) \to (x^\lambda, y^i, a^i_\lambda, \tfrac{1}{2}(D_\zeta a^i_\eta + D_\eta a^i_\zeta + a^j_\zeta D_j a^i_\eta + a^j_\eta D_j a^i_\zeta)).$$

We conclude that $\Gamma(\mathbf{H}^S) = \Gamma(\mathbf{H})^*$. $\square$

We remark that the solutions to the equation of the horizontal lifting, in the sense of Stratonovich $\circ \mathrm{d}\tilde{X}_t = H_{\tilde{X}_t} \circ \mathrm{d}X_t$, are the same to the solutions of $\mathrm{d}_2\tilde{X}_t = H^S_{\tilde{X}_t}\, \mathrm{d}_2 X_t$.

Next we give a characterization of 2-connections for a fiber bundle $\pi : P \to M$ in terms of their restriction to the tangent space and a section of $\mathrm{Ker}(\pi_*) \otimes \bigodot^2 T^*M \to P$.

Let $\pi : P \to M$ be a fiber bundle and $\mathbf{H}$ a 2-connection, it follows that $\mathbf{H}$ induces a section $C(\mathbf{H})$ of $\mathrm{Ker}(\pi_*) \otimes \bigodot^2 T^*M \to P$. Let $A, B \in T_{\pi(p)}M$, we set

$$C(\mathbf{H})_p(A, B) = (H_p - (H_R)^S_p) \circ Q^{-1}_{\pi(p)}(A \odot B).$$

This definition is independent of the representatives. In fact, let $G, J \in \tau_{\pi(p)}M$ such that $Q_{\pi(p)}(G) = Q_{\pi(p)}(J) = A \odot B$, this implies that $G = J + S$ with $S \in T_{\pi(p)}M$. Since $(H_p - (H_R)^S_p)/T_{\pi p}M = 0$ it follows that $(H_p - (H_R)^S_p)(G) = (H_p - (H_R)^S_p)(J)$. It remains to shows that $C(\mathbf{H})_p(A, B) \in \mathrm{Ker}(\pi_*(p))$, but it is clear since $\mathbf{H}$ and $\mathbf{H}^S$ are 2-connections. We have thus proved the following characterization of 2-connections.

Proposition 3. *Let $\pi : P \to M$ be a fiber bundle. The set of 2-connections are in bijection with the set of pairs (Γ, Σ) where Γ is a connection and Σ is a section of* $\mathrm{Ker}(\pi_*) \otimes \bigodot^2 T^*M \to P$.

Let $P(M, G)$ be a principal fiber bundle and $\mathbf{T}$ a connection in the sense of principal bundles. We set

$$C_2(\mathbf{T}) = \{\mathbf{H} : \text{is a principal 2-connection and } \mathbf{H}_R = \mathbf{T}\}.$$

We will show that $C_2(\mathbf{T})$ is an affine space associated with the vector space $A^e_{h,0}(P)$ of horizontal equivariant $\mathcal{G}$-valued forms $\phi : TP \odot TP \to \mathcal{G}$ such that

$$\phi \circ R_{g*} \otimes R_{g*} = \mathrm{Ad}(g^{-1}) \circ \phi, \qquad \phi/\mathrm{Ker}(\pi_* \otimes \pi_*) = 0.$$

In fact, we have that the action $\bullet$ of $A^e_{h,0}(P)$ on $C_2(\mathbf{T})$ is given by

$$\phi \bullet \mathbf{H} = \{\phi \bullet H_p = \gamma_p \circ \tilde{\phi}_p + H_p : p \in P\},$$

where $\gamma_p(A) = \frac{\mathrm{d}}{\mathrm{d}t} p \exp(tA)_{t=0}$ and $\tilde{\phi}_p : \tau_{\pi p}M \to \mathcal{G}$ is the unique morphism such that $\tilde{\phi}_p \circ \pi_*(p) = \phi_p \circ Q_p$.

Proposition 4. *Let $P(M, G)$ be a principal fiber bundle and $\mathbf{T}$ be a connection in the sense of principal fiber bundles. Then $(C_2(\mathbf{T}), \bullet)$ is the affine space associated to $A^e_{h,0}(P)$.*

Proof. It is clear from the definition of $\bullet$ that $(\Phi + \Psi) \bullet \mathbf{H} = \Phi \bullet (\Psi \bullet \mathbf{H})$ and $0 \bullet \mathbf{H} = 0$. We only need to show that $\bullet\mathbf{H} : A^e_{h,0}(P) \to C_2(\mathbf{T})$ is a bijection. Let $\mathbf{N} \in C_2(\mathbf{S})$, by the isomorphism theorem there exists an unique linear mapping $\Phi_p : T_pP \odot T_pP \to \mathcal{G}$ such that $\gamma_p \circ \Phi_p \circ Q_p = (N - H)_p \circ \pi_*(p)$, because $\mathrm{Ker}(Q_p) = T_pP \subset \mathrm{Ker}(\gamma_p^{-1} \circ (N - H)_p \circ \pi_*(p))$. We claim that $\Phi \in A^e_{h,0}(P)$ and $\Phi \bullet H_p = N_p$. In fact, let $A \in \mathrm{Ker}\,\pi_*(p) \odot \mathrm{Ker}\,\pi_*(p)$, we know that there exists $L \in \tau_pP$ such that $A = Q_pL$ and $\pi_*(p)L \in T_{\pi p}M$. Then

$$\Phi_p(A) = \Phi_p(Q_pL) = \gamma_p^{-1} \circ (N - H)_p \circ \pi_*(p)(L) = \gamma_p^{-1}(0) = 0.$$

Thus $\Phi_p/\mathrm{Ker}\,\pi_*(p) \odot \mathrm{Ker}\,\pi_*(p) = 0$. Let $A \in T_pP \odot T_pP$ and $g \in G$, we have that

$$\begin{aligned}
\Phi_p \circ (R_{g*} \odot R_{g*})(A) &= \Phi_{pg} \circ Q_{pg}(R_{g*}L) = \gamma_{pg}^{-1} \circ (N - H)_{pg} \circ \pi_*(pg)(R_{g*}L) \\
&= \gamma_{pg}^{-1} \circ (N-H)_{pg} \circ \pi_*(p)(L) = \gamma_{pg}^{-1} \circ R_{g*}((N - H)_p \circ \pi_*(p)(L)) \\
&= \mathrm{Ad}(g^{-1})(\gamma_p^{-1} \circ R_{g*}((N - H)_p \circ \pi_*(p)(L))) = \mathrm{Ad}(g^{-1})\Phi_p(A).
\end{aligned}$$

Thus $\Phi \in A^e_{h,0}(P)$ and it is clear that Φ is the unique element which satisfies $\Phi \bullet \mathbf{H} = \mathbf{N}$. $\square$

3. 2-Connections in vector bundles

Throughout this section we let $E = E(P, M, G, F, \lambda)$ denote a vector bundle associated to the principal bundle $P(M, G)$ obtained by the representation λ of G on the typical fiber F.

Let $\mathbf{H}$ be a principal 2-connection for $P(M, G)$. Naturally, $\mathbf{H}$ induces a 2-connection $\mathbf{H}^E = \{H_e^E : e \in E\}$ for $\pi : E \to M$. In fact, take $e \in \pi_E^{-1}(x)$, $p \in \pi_P^{-1}(x)$ and $\xi \in F$ such that $p\xi = e$, we define

$$H_e^E = (F_\xi)_*(p) \circ H_p,$$

where $F_\xi : P \to E$ is given by $F_\xi(q) = q\xi$. It follows easily that H_e^E is independent of the choice of the pair $(p, \xi) \in P \times F$ such that $p\xi = e$. We call $\mathbf{H}^E$ the induced 2-connection.

The purpose of this section is to develop the theory of 2-connections induced for vector bundles. There are many concepts associated to second order connections in vector bundles (see for instance [4,7,15,23,25,34]), we are particularly interested in one introduced by Liebermann [25].

For $k = 1, 2$ we have that J^2E is a vector bundle over M (see [21,29,34]). Let $J_0^k E_x$ be defined by

$$J_0^k E_x = \{j_x^k s : s \text{ is a local section of } E \text{ such that } s(x) = 0\}.$$

By setting $J_0^k E = \bigcup_{x \in M} J_0^k E_x$ we obtain a vector bundle over M. This vector bundle naturally embeds in $J^k E$ by the inclusion $j : J_0^k E \to J^k E$ which is an injective morphism of vector bundles. It follows that the sequence of vector bundles

$$0 \to J_0^k E \to J^k E \to E \to 0,$$

with the morphisms $j : J_0^k E \to J^k E$ and $\beta : J^k E \to E$ is exact.

Definition 4. We say that a splitting $\varphi : E \to J^k E$ of the above exact sequence is an L-connection of order k.

L-connections of order k are the connections in the sense of Liebermann [25].

Proposition 5. *Let $\mathbf{H}$ be a principal 2-connection for $P(M, G)$. Then $\Gamma(\mathbf{H}^E)$ is an L-connection of order 2.*

Proof. It is enough to show that $\Gamma(\mathbf{H}^E) : E \to J^2E$ is linear. Let $p \in P$, $f, g \in \pi_E^{-1}(x = \pi p)$ and $a \in \mathbb{R}$. We have that there exists $\zeta, \eta \in F$ such that $f = p\zeta$ and $g = p\eta$. Let s be a local section of P such that $H_p = s_*(x)$. Then

$$\begin{aligned}\Gamma(\mathbf{H}^E)(f + ag) &= j_x^2(F_{\zeta+a\eta} \circ s) = j_x^2(F_\zeta \circ s) + j_x^2(F_{a\eta} \circ s) \\ &= \Gamma(\mathbf{H}^E)(f) + a\Gamma(\mathbf{H}^E)(g). \quad \square\end{aligned}$$

We observe that every L-connection of order 2 induces an L-connection of order 1. In fact, let λ be an L-connection of order 2, we have that $\rho_1^2 \circ \lambda$ is an L-connection of order 1. We recall that $\rho_1^2 : J^2E \to J^1E$ is given by $\rho_1^2(j^2 s(x)) = j^1 s(x)$.

Let E be a vector space and λ be an L-connection of order 1 for E. We set

$$S_2(\lambda) = \{\mu \in S_2(E) : \rho_1^2 \circ \mu = \lambda\}.$$

We claim that $S_2(\lambda)$ have a natural structure of affine space. In order to display its associated vector space, we recall the so called *fundamental sequence* which states the sequence of vector bundles

$$0 \to E \otimes (TM \odot TM)^* \to J^2E \to J^1E \to 0$$

with the morphisms $\tau : E \otimes (TM \odot TM)^* \to J^2E$ (the fundamental identity) and $\rho_1^2 : J^2E \to J^1E$ is exact (see for instance [22]).

Lemma 3. *$S_2(\lambda)$ is an affine space associated to vector space of sections of the vector bundle*

$$\mathrm{Hom}((TM \odot TM), \mathrm{Hom}(E, E)).$$

Proof. Take μ and η in $S_2(\lambda)$, since $\rho_1^2(\mu) = \rho_1^2(\eta)$ we have that $\mu - \eta \in \mathrm{Ker}\, \rho_1^2$. By the fundamental sequence, it follows that there exists an unique γ in $E \otimes (TM \odot TM)^*$ such that $\tau(\gamma) = \mu - \eta$. We can consider $\mu - \eta$ as a section of $\mathrm{Hom}((TM \odot TM), \mathrm{Hom}(E, E))$. □

Let E be a vector bundle, we denote by BE the principal fiber bundle of frames of E. The next theorem shows that every L-connection of order 2 is induced by a unique principal 2-connections for BE.

Theorem 3. *Let E be a vector bundle and $\varphi : E \to J^2E$ be an L-connection of order 2. Then there exists an unique 2-connection $\mathbf{H}$ for BE such that $\varphi = \Gamma(\mathbf{H}^E)$.*

Proof. Let $p = (e_1, \dots, e_n) \in BE$, we have that there exist $s^1, \dots, s^n$ local sections of E such that $\varphi(e_i) = j^2_{\pi(p)}s^i$ for $i = 1, \dots, n$. We define

$$H_p = (s^1, \dots, s^n)_*(\pi(p)) : \tau_{\pi(p)}M \to \tau_p BE.$$

It is easy to check that $\mathbf{H} = \{H_p : p \in P\}$ is a principal 2-connection for BE. Let $p = (e_1, \dots, e_n) \in BE$, $\xi = (\xi^1, \dots, \xi^n) \in F$ and $e = p\xi$. It follows that

$$H_e^E = (F_\xi)_*(p) \circ H_p = (s^1\xi^1 + \cdots + s^n\xi^n)_*(\pi(p))$$

thus

$$\Gamma(\mathbf{H}^E)(e) = j^2_{\pi(p)}(s^1\xi^1 + \cdots + s^n\xi^n) = \varphi(e_1)\xi^1 + \cdots + \varphi(e_n)\xi^n = \varphi(e).$$

It remains to prove the uniqueness of $\mathbf{H}^E$. Let $\mathbf{S}$ and $\mathbf{T}$ be principal 2-connections for BE such that $\Gamma(\mathbf{S}^E) = \Gamma(\mathbf{T}^E)$. Since $\mathbf{S}_R = \mathbf{T}_R$, it follows that there exists Ψ such that $\Psi \bullet \mathbf{S} = \mathbf{T}$. But $\Psi_p = 0$ for all $p \in BE$ since $(F_\xi)_*(p) \circ \gamma_p \circ \tilde{\Psi}_p = 0$ for all $p \in BE$ and $\xi \in F$. This proves the theorem. □

Let E be a vector bundle, φ be an L-connection of order 2, $e \in E$ and X be an M-valued semimartingale such that $X_0 = \pi(e)$. Let $p \in BE$ and $\xi \in F$ such that $e = p\xi$. We can define a parallel transport of e along X with respect to φ as

$$//^{\varphi}_{X_t}(e) = \tilde{X}(p)_t\xi,$$

where $\tilde{X}(p)$ is the SHL of X starting at p with respect to the principal 2-connection $\mathbf{H}$ given by the above theorem. It is clear that $//^{\varphi}_{X_t}(e)$ is independent of the representatives $p \in BE$ and $\xi \in F$. In fact, let $q \in BE$ and $\zeta \in F$ such that $e = q\zeta$. We observe that there exists $g \in GL(F)$ such that $q = pg^{-1}$ and $\zeta = g\xi$, thus $\tilde{X}(q) = R_{g^{-1}}\tilde{X}(p)$ is the SHL of X in q. We conclude that $\tilde{X}(q)\zeta = \tilde{X}(p)\xi$.

Obviously, $//^{\varphi}_{X_t(\omega)} : E_x \to E_{X_t(\omega)}$ is a linear isomorphism.

Example 2. The correspondence between extensions of ordinary parallel transport on vector bundles to semimartingales and prolongations of the connections on the base space to the total space, was discovery by Meyer [27], Arnaudon and Thalmaier [2] extend the work of Meyer and apply it to different geometric situations. We will show how extensions of the ordinary parallel transport are in correspondence with L-connections.

Let M be a manifold equipped with a connection ∇ and let $\pi : E \to M$ be a vector bundle over M equipped with a covariant derivative operator ∇^E, they study some connections ∇' on E obtained as prolongations of the pair (∇, ∇^E) such that $\pi : E \to M$ is affine. Examples of such prolonged connections are given by the horizontal lift, and by the complete lift in the case $E = TM$.

We note that the prolonged connection ∇' induce naturally an L-connection $\varphi^{\nabla'}$. In fact, the covariant derivative operator ∇^E on E gives a splitting of the tangent bundle TE into the horizontal bundle HE and the vertical bundle VE. Let $h_e = (\pi_*(e)|H_eE)^{-1} : T_{\pi(e)} \to H_eE$ be the horizontal lift. From Emery (see [12, Lemma 11, pp. 426]), we have that there exist a unique 2-connection on E, $\mathbf{H}' = \{H'_e : \tau_{\pi(e)}M \to \tau_e E : e \in E\}$ such that $H'_e|T_{\pi(e)}M = h_e$ and H_e is semi-affine for $e \in E$. It is easy to check that $\varphi^{\nabla'} := \Gamma(\mathbf{H}')$ is an L-connection.

Finally, we observe that if X is an M-valued semimartingale and $e \in E_{X_0}$, $//^{\nabla'}_{X_t}(e)$ the parallel transport of e along X with respect to $\varphi^{\nabla'}$ verifies the Itô equation:

$$\mathrm{d}^{\nabla'} //^{\nabla'}_{X_t}(e) = h_{//^{\nabla'}_{X_t}(e)}(\mathrm{d}^{\nabla} X_t)$$

with the initial condition $//^{\nabla'}_{X_t}(e) = e$.

Example 3. Let $E = E(M, \rho, F)$ be a vector bundle associated to the principal fiber bundle $P(M, G)$ with fiber F, $\mathbf{H} = \{H_p : p \in P\}$ be a connection for $P(M, G)$ and ∇^E the covariant derivative operator induced by $\mathbf{H}$. Let ∇ be a G-invariant covariant derivative operator of P such that $\nabla_A B$ is vertical for any vertical field A. Let $X(x)$ be the solution starting at $x \in M$ of the Stratonovich equation on M:

$$\mathrm{d}X_t(x) = A_0(X_t(x))\,\mathrm{d}t + \sum_{i=1}^{n} A_i(X_t(x)) \circ \mathrm{d}B^i_t,$$

where $A_0, \ldots, A_n$ are vector fields on M and $(B^1_t, \ldots, B^n_t)$ a Brownian motion on $\mathbb{R}^n$. According to the comments after Lemma 2, we have that there exists a unique principal 2-connection $\mathbf{H}^\nabla := \{H^\nabla_p : p \in P\}$ (prolongation of $\mathbf{H}$) satisfying (2). Let $Y(p)$ be the SHL with respect to $\mathbf{H}^\nabla$ of X starting at $p \in P$ ($\pi(p) = x$). We know that $Y(p)$ verifies the following Stratonovich equation on M (see [5] for details):

$$\mathrm{d}Y_t(p) = \left(HA_0 - \frac{1}{2}\sum_{i=1}^{n} \omega^{\mathbf{H}}(\nabla_{HA_i} HA_i)^* \right)(Y_t(p))\,\mathrm{d}t + \sum_{i=1}^{n} HA_i(Y_t(p)) \circ \mathrm{d}B^i_t.$$

Applying the results of Akiyama (see [1 pp. 86]) to the above formula for the SHL, we obtain an Itô formula for the parallel transport $//^{\nabla}_t(\pi(p)) = Y_t(p) \circ p^{-1} : E_{\pi(p)} \to E_{X_t(\pi(p))}$ induced by $\mathbf{H}^\nabla$ on E. Let σ be a section of E, we have that

$$\mathrm{d}(//^{\nabla}_t)^{-1}\sigma(X_t) = (//^{\nabla}_t)^{-1} \left(\nabla^E_{A_0} + \frac{1}{2}\sum_{i=1}^{n} \left((\nabla^E_{A_i})^2 - \frac{1}{2}\omega^H(\nabla_{HA_i} HA_i) \right) \right) \sigma(X_t)\,\mathrm{d}t$$
$$+ \sum_{i=1}^{n} ((//^{\nabla}_t)^{-1}(\nabla^E_{A_i})\sigma(X_t))\,\mathrm{d}B^i_t.$$

Now, we extend the covariant derivative operators to second order.

Definition 5. Let E be a vector bundle over M. A 2-covariant derivative operator for E (2-CDO for E) is a mapping ∇ which assigns to each pair of sections L of τM and ϕ of E a section ∇_L^{ϕ} of E, which satisfies the following properties:

(i) $\nabla_{L+T}\phi = \nabla_L\phi + \nabla_T\phi$,
(ii) $\nabla_{fL}\phi = f\nabla_L\phi$,
(iii) $\nabla_L(\phi+\psi) = \nabla_L\phi + \nabla_L\psi$ and
(iv) $\nabla_L f\phi = L(f)\phi + f\nabla_L\phi + 2\nabla_{Q(L,f)}\phi$

for any sections L, T of τM, ϕ, ψ of E and $f \in C^\infty(M)$. We call $\nabla_L\phi$ the 2-covariant derivative of ϕ in the direction of L.

Remark 1. Every 2-covariant derivative operator ∇ induces a covariant derivative operator ∇^R by restriction to sections of *TM*. We say that ∇^R is induced by ∇.

In order to construct 2-CDO, we recall some facts about sections in associated vector bundles (see [20,21]). Let $P(M, G)$ be a principal fiber bundle and $E = E(P, M, G, F, \lambda)$ be an associated vector bundle. We can associate with each section σ of E a function $F[\sigma] : P \to F$ defined by $F[\sigma](p) = p^{-1}\sigma(\pi p)$ where each $p \in P$ is regarded as a linear mapping $p : F \to E_{\pi p}$. Notice that $F[\sigma]$ satisfies $F[\sigma] \circ R_g = \lambda_{g^{-1}} \circ F[\sigma]$ for all $g \in G$. Conversely, if a function $f : P \to F$ satisfies $f \circ R_g = \lambda_{g^{-1}} \circ f$ for all $g \in G$, we have a section of E, $S[f](\pi(p)) = pf(p)$ for all $p \in P$. Obviously, $F[S[f]] = f$ and $S[F[\sigma]] = \sigma$.

Proposition 6. *Let* $\mathbf{H}$ *be a principal 2-connection for* $P(M, G)$. *Then* $\nabla(\mathbf{H})$ *defined by*

$$\nabla(\mathbf{H})_L\phi(\pi(p)) = S[(H_pL)F[\phi]]$$

is a 2-CDO for E.

Proof. We observe that

$$Q(H_pL, f\circ\pi) = H_pQ(L, f)$$

for all sections L of τM and $f \in C^\infty(M)$. In fact, we have that

$$Q(H_pL, f\circ\pi) = Q(H_pL)(f\circ\pi) = (H_p\otimes H_p)(QL)(f\circ\pi),$$

since H_p is a Schwartz morphism. On the other hand, if $g \in C^\infty(P)$

$$\begin{aligned}((H_p\otimes H_p)(QL)(f\circ\pi))(g) &= (H_p\otimes H_p)(QL)(\mathrm{d}(f\circ\pi), \mathrm{d}g) = QL(H_p^*\mathrm{d}(f\circ\pi), H_p^*\,\mathrm{d}g)\\ &= QL(\mathrm{d}f\circ\pi_*\circ H_p, H_p^*\,\mathrm{d}g) = QL(\mathrm{d}f, H_p^*\,\mathrm{d}g)\\ &= H_pQ(L, f)(g).\end{aligned}$$

Thus $(H_p\otimes H_p)(QL) = H_pQ(L, f)$, then $Q(H_pL, f\circ\pi) = H_pQ(L, f)$.

We have that $\nabla(\mathbf{H})$ is well defined. In fact

$$\begin{aligned}F[\nabla(\mathbf{H})_L\phi]\circ R_g(p) &= F[\nabla(\mathbf{H})_L\phi](pg) = H_{pg}(L)F[\phi] = ((R_g)_*H_pL)F[\phi]\\ &= H_pL(F[\phi]\circ R_g) = H_pL(\lambda_{g^{-1}}\circ F[\phi]) = \lambda_{g^{-1}}\circ(H_pL)F[\phi]\\ &= \lambda_{g^{-1}}\circ F[\nabla(\mathbf{H})_L\phi],\end{aligned}$$

which implies that $\nabla(\mathbf{H})_L\phi$ is a section of E.

It is easy to check that $\nabla(\mathbf{H})$ satisfies the properties defining a 2-CDO. For example, item (iv) is obtained by

$$\begin{aligned} F[\nabla(\mathbf{H})_L f\phi](p) &= (H_pL)(f\circ\pi)F[\phi] \\ &= ((H_pL)f\circ\pi)F[\phi] + f\circ\pi(H_pL)F[\phi] + 2Q(HL, f\circ\pi)_pF[\phi] \\ &= (H_pL(f\circ\pi))F[\phi] + f\circ\pi(H_pL)F[\phi] + 2H_pQ(L,f)F[\phi] \\ &= F[Lf\phi + f\nabla(\mathbf{H})_L\phi + 2\nabla(\mathbf{H})_{Q(L,f)}\phi](p). \quad \square \end{aligned}$$

It is also possible to construct an analogous to the Stratonovich prolongation for covariant derivative operators. In fact, given ∇ a CDO for the vector bundle E, it is easy to prove that there exists ∇^S the unique 2-CDO for E such that

1. ∇ is induced by ∇^S,
2. $\nabla^S_{\{X,Y\}}\phi = \{\nabla_X, \nabla_Y\}\phi$for any sections X, $Y$of TM and $\phi$of$E$.

We call ∇^S the Stratonovich prolongation of ∇.

Let λ be an L-connection of order 1, then $\lambda^S = \mathrm{Sym}\circ j^1\lambda\circ\lambda$ is an L-connection of order 2, called the Stratonovich prolongation of λ.

Given a vector bundle E, there is an affine isomorphism between the connections of BE and covariant derivative operators of E (see for instance [20]). The generalization for 2-CDO is established by our next theorem.

Theorem 4. *The map ∇ from the set of 2-connections for BE to the set of 2-CDO for E is an affine isomorphism.*

Proof. We see at once that ∇ is an affine morphism. Let Υ be the linear map from $A^e_{h,0}(BE)$ to sections of $\mathrm{Hom}(\tau M, \mathrm{Hom}(E, E))$ defined by

$$\Upsilon(\phi)(L)(e) = -p(\tilde{\phi}_pL)p^{-1}e,$$

where $e \in E$, $p \in BE$ such that $\pi(p) = e$ and L is a section of $\tau_{\pi p}M$. We claim that Υ is an isomorphism. Clearly, Υ is linear and injective. Let a be a section of $\mathrm{Hom}(\tau M, \mathrm{Hom}(E, E))$, we observe that $\Upsilon(\phi) = a$ where $\phi_p(Q_p(HL)) = -p^{-1}a(L)p$ for $L \in \tau_{\pi p}M$, this shows the claim. Let $\mathbf{H}$ be a 2-connection for BE, we have that

$$\nabla(\mathbf{H}) = \nabla(\mathbf{H}^S_R) + \Upsilon(\mathbf{H} - \mathbf{H}^S_R),$$

which proves the theorem. $\square$

Remark 2. Note that 2-covariant derivative operators on vector bundles induce 2-covariant derivative operators on tensor products and wedge products in a straightforward way and the parallel transport obtained is the tensor product, respectively, wedge product of the given transports. A 2-covariant derivative operator is also canonically induced on the dual bundles and the associated parallel transport is the dual of the inverse parallel transport in the vector bundle.

Let E be a vector bundle, ∇ be a 2-CDO for E, $e \in E$ and X be an M-valued semimartingale such that $X_0 = \pi(e)$. Let $p \in BE$ and $\xi \in F$ such that $e = p\xi$. We can define a parallel transport of e along X with respect to ∇ by

$$//^{\nabla}_{X_t}(e) = \tilde{X}(p)_t\xi$$

where $\tilde{X}(p)$ is the SHL of X starting at p with respect to the principal 2-connection $\mathbf{H}$ given by the above theorem.

Let L be a section of τM. We recall that a continuous M-valued semimartingale X, is an L-diffusion if for all $f \in C^{\infty}(M)$, we have that

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,\mathrm{d}s$$

is a local martingale.

Proposition 7. *Let ∇ a 2-CDO for the vector bundle E, L be a section of τM and ϕ be a section of E. Then*

$$(//^{\nabla}_{X_t})^{-1}\phi \circ X_t = (local\,martingale)_t + \phi(x) + \int_0^t //^{-1}_{X_s}(\nabla_L \phi \circ X_s)\,\mathrm{d}s,$$

where X is an L-diffusion with $X_0 = x$.

Proof. We have that

$$\begin{aligned}
(//^{\nabla}_{X_t})^{-1}\phi \circ X_t &= p\tilde{X}_t^{-1}(\phi \circ X_t) = pF[\phi](\tilde{X}_t) \\
&= p((\text{local martingale})_t + F[\phi](p) + \int_0^t HL(F[\phi])(\tilde{X}_s)\,\mathrm{d}s) \\
&= p((\text{local martingale})_t + F[\phi](p) + \int_0^t F[\nabla_L \phi](\tilde{X}_s)\,\mathrm{d}s) \\
&= (\text{local martingale})_t + \phi(\pi(p)) + p\left(\int_0^t F[\nabla_L \phi](\tilde{X}_s)\,\mathrm{d}s\right) \\
&= (\text{local martingale})_t + \phi(\pi(p)) + p\left(\int_0^t (\tilde{X}_s)^{-1}\nabla_L \phi(X_s)\,\mathrm{d}s\right) \\
&= (\text{local martingale})_t + \varphi(\pi(p)) + \int_0^t //^{-1}_{X_s}(\nabla_L \phi \circ X_s)\,\mathrm{d}s. \quad \square
\end{aligned}$$

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